A Proof Sketch Of Something Which May Possibly Be A Conjecture of Oege de Moor

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This note puports to prove something which Oege de Moor presented as an open problem in a talk entitled "Pointwise Relations" at the Computer Laboratory on December 1st. Since I've rephrased everything in terms with which I'm more familiar[1] (and may well have misunderstood or misremembered what he said), it's entirely possible that it doesn't, however.

Oege starts with a simply-typed lambda calculus. This is given two interpretations, one in $\mathbb{S}$et and one in $\mathbb{R}$el. Now $\mathbb{R}$el is the Kleisli category of the powerset monad $\mathbb{P}$ on $\mathbb{S}$et and I believe that Oege's direct relational semantics is the same one as you get by factoring through the call-by-value translation into Moggi's computational metalanguage and then interpreting that in $\mathbb{S}$et with $T = \mathbb{P}$. The call-by-value translation has the following shape:

$$(\Gamma \vdash M : A)^* \quad = \quad \Gamma^* \vdash M^* : T(A^*)$$

where

$$\begin{array}{rcl}
\text{Types} & & \\
G^* & = & G \quad G \text{ a ground type} \\
(A \times B)^* & = & A^* \times B^* \\
(A \to B)^* & = & A^* \to T(B^*)
\end{array}$$

$$\begin{array}{rcl}
\text{Terms in Context} & & \\
(\Gamma, x : A \vdash x : A)^* & = & \Gamma^*, x : A^* \vdash \mathrm{val}\, x : T(A^*) \\
(\Gamma \vdash (M\ N) : B)^* & = & \Gamma^* \vdash (\mathrm{let}\, x \Leftarrow M^* \,\mathrm{in}\, (\mathrm{let}\, y \Leftarrow N^* \,\mathrm{in}\, x\ y)) : T(B^*) \\
(\Gamma \vdash (\lambda x : A.M) : A \to B)^* & = & \Gamma^* \vdash \mathrm{val}\, (\lambda x : A^*.M^*) : A^* \to T(B^*)
\end{array}$$

The $\mathrm{val}\,(\cdot)$ form is interpreted by the unit of the monad and $\mathrm{let}\,\cdot \Leftarrow \cdot\,\mathrm{in}\,\cdot$ by Kleisli composition.

I don't think that what follows depends on anything that's very specific to $\mathbb{S}$et or the powerset monad, but I haven't got around to rewriting it in an element-free way in terms of CCCs with relations and seeing just what the

[1] "Mathematicians are like Frenchmen: whatever you say to them, they translate into their own language and forthwith it is something entirely different." – Goethe.

conditions are. Not only will I be frightfully uncategorical, but I'll also confuse syntax and semantics all over the place, confident that the $\imath$'s can be dotted and the ℓ's crossed if there's any interest ...

We start by defining a relation $\mathcal{R}_A$ between (the intepretations of) A and A^* for each type A of the source language. To deal with the fact that we've got computation types around, we'll also need a trivial auxiliary relation $\mathcal{R}_A^T$ which relates A with $T(A^*)$:

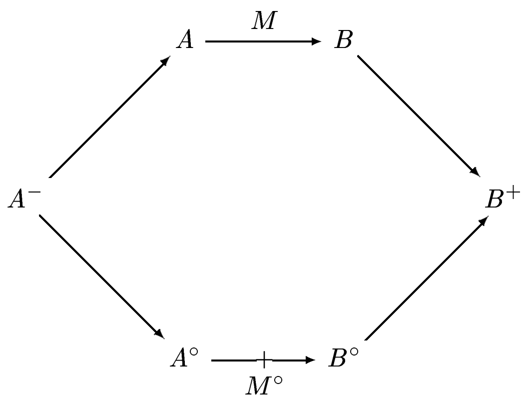
$$
\begin{array}{rcl}
x \; \mathcal{R}_G \; y & \iff & x = y \\
f \; \mathcal{R}_{A \to B} \; g & \iff & \forall x \in A, y \in A^*. x \; \mathcal{R}_A \; y \Rightarrow (f\;x) \; \mathcal{R}_B^T \; (g\;y) \\
x \; \mathcal{R}_A^T \; y & \iff & \exists y' \in A^*. (y = \mathrm{val}\, y') \wedge (x \; \mathcal{R}_A \; y')
\end{array}
$$

(Probably hiding 'η is mono' in the computation type case.) A simple induction on terms in context yields the usual "fundamental theorem of logical relations":

Lemma 1. *If* $x_1 : A_1, \dots x_n : A_n \vdash M : B$ *and for all* $1 \leq i \leq n \vdash V_i : A_i$, $\vdash W_i : A_i^*$, *and* $V_i \; \mathcal{R}_A \; W_i$, *then* $M[V_i/x_i] \; \mathcal{R}_A^T \; M^*[W_i/x_i]$. □

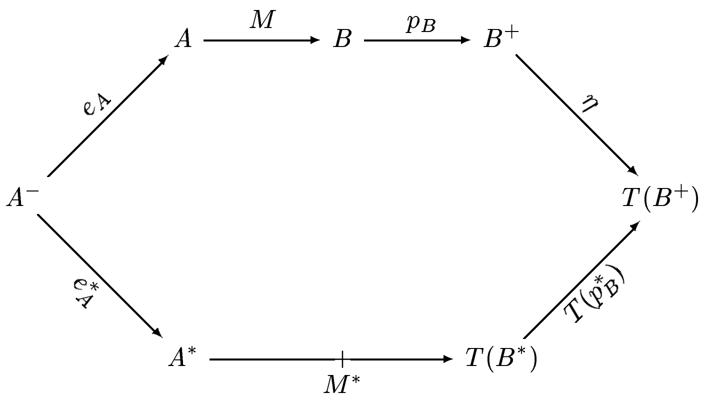
The above should be read with semantic brackets in appropriate places and probably with W and V being elements of the model rather than terms (and thus composition instead of substitution), but it doesn't make any difference.

But Oege's theorem actually looked something like this:



Where A° and B° are the relational interpretations of the types A and B, M° is the relational interpretation of the term M with one free variable. The $(\cdot)^+$ and $(\cdot)^-$ are inductively defined translations which replace function spaces in the original type by 'relation spaces' in all positive (resp. negative) positions. There are canonical coercion functions $A^- \to A$, $A \to A^+$, $A^- \to A^\circ$ and $A^\circ \to A^+$ which are defined in the 'obvious' way. Note that it's relational composition along the bottom of the diagram.

What does that look like in terms of explicit computational types? I confidently assert (but am too lazy to check) that it's this:



Where

$$\begin{aligned} G^+ &= G \\ G^- &= G \\ (A \to B)^+ &= A^- \to T(B^+) \\ (A \to B)^- &= A^+ \to B^- \end{aligned}$$

and

$$\begin{aligned} e_G(g) &= g \\ p_G(g) &= g \\ e^*_G(g) &= g \\ p^*_G(g) &= g \\ e_{A\to B}(f) &= e_B \circ f \circ p_A \\ p_{A\to B}(f) &= \eta \circ p_B \circ f \circ e_A \\ e^*_{A\to B}(f) &= \eta \circ e^*_B \circ f \circ p^*_A \\ p^*_{A\to B}(f) &= T(p^*_B) \circ f \circ e^*_A \end{aligned}$$

I claim that this is implied by Lemma 1, which requires me to connect the logical relation and all those funny es and ps:

Proposition 2. *For any type A*

1. $\forall x \in A^-.\ e_A(x)\ \mathcal{R}_A\ e^*_A(x)$;
2. $\forall x \in A, y \in A^*.\ x\ \mathcal{R}_A\ y \Rightarrow p_A(x) = p^*_A(y)$.

Proof. The two parts are proved simultaneously by induction on A. The base case is trivial, whilst for function types we reason as follows:

1. If $f \in (A \to B)^-$, we want to know that $e_{A\to B}(f)\ \mathcal{R}_{A\to B}\ e^*_{A\to B}(f)$. Expanding the definitions that's

$$(e_B \circ f \circ p_A)\ \ \mathcal{R}_{A\to B}\ \ (\eta \circ e^*_B \circ f \circ p^*_A)$$

By the definition of $\mathcal{R}_{A\to B}$ that means we have to show that for any a, b with $a\,\mathcal{R}_A\,b$

$$(e_B \circ f \circ p_A)(a) \;\; \mathcal{R}_B^T \;\; (\eta \circ e_B^* \circ f \circ p_A^*)(b)$$

By induction (second part), we know $p_A(a) = p_A^*(b)$ so the above is

$$e_B(f(p_A(a))) \;\; \mathcal{R}_B^T \;\; \eta(e_B^*(f(p_A(a))))$$

By the definition of $\mathcal{R}_B^T$ (definitely *do* want η mono) that holds if

$$e_B(f(p_A(a))) \;\; \mathcal{R}_B \;\; e_B^*(f(p_A(a)))$$

which holds by induction (first part).

2. Now assume $f\,\mathcal{R}_{A\to B}\,g$ and we want $p_{A\to B}(f) = p_{A\to B}^*(g)$. That's

$$(\eta \circ p_B \circ f \circ e_A) = (T(p_B^*) \circ g \circ e_A^*)$$

so pick an arbitrary $x \in A^-$, then we need show

$$(\eta \circ p_B \circ f \circ e_A)(x) = (T(p_B^*) \circ g \circ e_A^*)(x) \qquad (1)$$

By induction (first part), we know $(e_A\,x)\;\mathcal{R}_A\;(e_A^*\,x)$ and hence, as f and g are related, $(f(e_A\,x))\;\mathcal{R}_B^T\;(g(e_A^*\,x))$. By the definition of $\mathcal{R}_B^T$, that means $g(e_A^*\,x) = \eta(v)$ for some v such that $(f(e_A\,x))\;\mathcal{R}_B\;v$. But then

$$\begin{array}{rcl} T(p_B^*)(g(e_A^*\,x)) & = & T(p_B^*)(\eta\,v) \\ & = & \eta(p_B^*\,v) \quad \text{(monad defn.)} \end{array}$$

So we can establish Equation 1 if we can show

$$(p_B(f(e_A\,x))) = (p_B^*\,v)$$

which follows immediately from the fact that $(f(e_A x))\;\mathcal{R}_B\;v$ and induction (second part).

□

Now, look back at my version of Oege's diagram.

Corollary 3. *If $x : A \vdash M : B$ then*

$$e_A; [\![M]\!]; p_B; \eta \;\; = \;\; e_A^*; [\![M^*]\!]; T(p_B^*)$$

Proof. If $x \in A^-$ then $(e_A\;x)\;\mathcal{R}_A\;(e_A^*\;x)$ by part 1 of Proposition 2. Hence, by Lemma 1, $[\![M]\!](e_A\;x)\;\mathcal{R}_B^T\;[\![M^*]\!](e_A^*\;x)$. Hence we're done by part 2 of Proposition 2, just as we were in that proof. (Not suprising, since we're in a CCC.) □